

Hood on positive complete MC structures

Wednesday, April 6, 2016 10:33 AM + Why we need it.

Recall Up^G is the cat of orth G -spectra and G -maps.

A CGMC is a model category \mathcal{M} with generating sets $I + J$ of cofibrations and trivial cofibrations. I and J admit the small object argument (SOA).

cofibre $M = \text{LLP}(\text{RLP}(I)) =: \text{cofibre}(I)$

Kan Recognition Thm says when (\mathcal{M}, I, J) is a CGMC for homotopical cat \mathcal{M} .

$$\textcircled{1} \quad \mathcal{M} = \text{Top}, \quad I = \left\{ S^{n-1} \xrightarrow{\sim} D^n : n \geq 0 \right\}$$

$$J = \left\{ I^n \xrightarrow{\sim} I^n : n \geq 0 \right\}$$

$$\textcircled{2} \quad \mathcal{M} = \text{Up}^G \quad I = \left\{ G_+, \bigwedge_N S^{-V}_+ (S^{n-1}_+ \xrightarrow{\sim} D^n_+) : n \geq 0, H \subseteq G, V = \text{rep of } G \right\}$$

$$\textcircled{3} \quad J = \left\{ \quad \quad \quad (I^{n-1}_+ \xrightarrow{\sim} I^n_+) \quad \quad \quad \right\}$$

This is the complete strict model structure

cofibre (J) = levelwise relative J -cell complexes

$\text{RLP}(I)$ = levelwise trivial fibrations

$$\begin{array}{ccc} G_+ & S^{-V}_+ S^{n-1}_+ \xrightarrow{\sim} X & S^{-V}_+ S^{n-1}_+ \xrightarrow{\sim} i_H^G X & S^{n-1}_+ \xrightarrow{\sim} (i_{H+1}^G X)_V \\ \downarrow & \downarrow & \downarrow & \downarrow \\ G_{n+1} & S^{-V}_+ D^n_+ \xrightarrow{\sim} Y & S^{-V}_+ D^n_+ \xrightarrow{\sim} i_X^G X & D^n_+ \xrightarrow{\sim} (i_{N+1}^G X)_V \end{array}$$

We need a stable complete MC structure.

We need to invent $S^{(V \otimes W)} \xrightarrow{\sim} S^V \rightarrow S^W$

$$X = \varinjlim_V S^{-V}_+ X_V$$

The map $S^{-\infty \otimes W} \xrightarrow{\sim} S^V \xrightarrow{e_{V,W}} S^{-W}$ may not be a cofibre
so we factor it

$$S^{-\infty \otimes W} \xrightarrow{\sim} S^V \xrightarrow[\tilde{e}_{V,W}]{{\text{while}}} \widehat{S}^{V,W} \xrightarrow[\text{fibre}]{\text{true}} S^{-W}$$

Let $J = K \cup \left\{ \tilde{e}_{V,W} \right\}$ (really $\{I \sqcup \tilde{e}_{V,W}\} \cup K$)

We will see later that $RLP(J)$ relative \mathbb{S} -spectra

We get the stable complete MC structure.

We will need $S^G \xrightarrow[\cup]{\text{Sym}} \text{Comm}^G$ symmetrically
function

To satisfy Kan transfer thm. It does not
work for the
stable complete
MC structures

$$\text{Sym } X = \bigvee_{n \geq 0} \text{Sym}^n X$$

$$\text{Sym}^n X = X^{\wedge n} / \Sigma_n$$

We need $\text{Sym } I$, $\text{Sym } J$ to generate
CFMC structures in Comm^G .

We need for $A \xrightarrow{\cong} B$ in J ,

$$\begin{array}{ccc} \text{Sym } A & \xrightarrow{\cong} & \text{Sym } B \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y = \text{pushout} \end{array} \quad \text{in } \text{Comm}^G$$

$$\begin{array}{ccc} \text{KTT} & & \\ & \swarrow & \searrow \\ CFTC & \approx & \mathcal{M} \xrightleftharpoons[\cup]{F} \mathcal{N} \end{array}$$

Example $\zeta^{-1}, \zeta^1 \longrightarrow \zeta^0$

$$\text{Sym}^n \zeta^0 = (\zeta^0)^m / \Sigma_m = \zeta^0 / \Sigma_m = \zeta^0$$

$$\begin{aligned}\text{Sym}^n (\zeta^{-1}, \zeta^1) &= (\zeta^{-1}, \zeta^1)^m / \Sigma_m \\ &= (\zeta^{-n}, \zeta^n) / \Sigma_n\end{aligned}$$

Recall $(\zeta^n)_k = f(n, k)$ and $O(n)$ acts freely
so Σ_n

$$\begin{aligned}\text{so } (\zeta^{-n}, \zeta^n) / \Sigma_n &= (\zeta^{-n}, \zeta^n)_{h\Sigma_n} \cong (\zeta^0)_{h\Sigma_n} \\ &= E\Sigma_{n+1} \cap \zeta^0 = B\Sigma_{n+1} \cap \zeta^0\end{aligned}$$

Hence $\text{Sym} (\zeta^{-1}, \zeta^1) = \bigvee_{n \geq 0} B\Sigma_{n+1} \cap \zeta^0 \neq \zeta^0$



Thus we need to require $\dim V > 0$
in (*) to avoid this problem

In order to play nicely with \mathcal{O}^H
we need $\dim V^H > 0$.